Analysis of the Frank-Wolfe Method for Convex Composite Optimization involving a Logarithmically-Homogeneous Barrier

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Joint work with Robert M. Freund (MIT Sloan School of Management)

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- Assume $\mathsf{dom}\, F \neq \emptyset$, so at least one minimizer $x^* \in \mathsf{dom}\, F$ exists, and define $F^* := F(x^*)$.

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by solving a linear-optimization sub-problem.

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  - Update $x^{k+1} = (1-\alpha_k)x^k + \alpha^k v^k$.

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▷ FW has been generalized to the composite setting:

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in e.g., Bach (2015) and Nesterov (2018), where the subproblem becomes:

$$v^k \in \arg\min_{x\in\mathbb{R}^n} \langle \nabla f(\mathsf{A}x^k), \mathsf{A}x \rangle + h(x).$$

However, note that all of these works assume that f is L-smooth.

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- We identified the *logarithmic-homogeneity* as the key element in Khachiyan's analysis, and proposed a (generalized) FW method with adaptive step-size for the much broader problem class (P).
- Our complexity bound essentially recovers Khachiyan's result, and is affine-invariant (along with other desirable properties).

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  1. $\left|D^3 f(u)[w,w,w]\right| \leq 2(\langle H(u)w,w\rangle)^{3/2} \;\; \forall\, u \in \mathsf{int}\,\mathcal{K}, \forall\, w \in \mathbb{R}^m$,
  2. $f(u_k) \to \infty$ for any $\{u_k\}_{k\geq 1} \subseteq \mathsf{int}\,\mathcal{K}$ such that $u_k \to u \in \mathsf{bd}\,\mathcal{K}$,
  3. $f(tu) = f(u) - \theta \ln(t) \;\; \forall\, u \in \mathsf{int}\,\mathcal{K}, \forall\, t > 0$,

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A Motivating Example: D-optimal Design

$$\begin{aligned} \max_p \quad & h(p) \triangleq \ln\det\left(\textstyle\sum_{i=1}^m p_i a_i a_i^\top\right) \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^m p_i = 1,\ p_i \ge 0,\ \forall\, i \in [m]. \end{aligned} \tag{D-OPT}$$

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- We resolve this mystery and generalize his method to the much broader class of problems in (P), even while relaxing the exact line-search requirement.

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- For convenience, we also represent A in its matrix form $A \in \mathbb{R}^{N\times N}$, where $N := mn$, and vectorize Y and X into $y \in \mathbb{R}^N$ and $x \in \mathbb{R}^N$, respectively. Notation: we write $x = \mathsf{vec}(X)$ and $X = \mathsf{mat}(x)$, etc.

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Poisson Image Deblurring with TV Regularization, continued

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$$\begin{aligned} \min_{x\in\mathbb{R}^N} \quad & \bar{F}(x) := -\textstyle\sum_{l=1}^N y_l \ln(a_l^\top x) + (\sum_{l=1}^N a_l)^\top x + \lambda \mathrm{TV}(x) \\ \mathrm{s.\,t.} \quad & 0 \le x \le Me \ , \end{aligned} \qquad \texttt{(Deblur)}$$

- (`Deblur`) has a (standard) total-variation (TV) regularization term to recover a smooth image with sharp edges. The TV term is given by

$$\begin{aligned} \mathrm{TV}(x) := & \textstyle\sum_{i=1}^m \sum_{j=1}^{n-1} \left| [\mathsf{mat}(x)]_{i,j} - [\mathsf{mat}(x)]_{i,j+1} \right| \\ & + \textstyle\sum_{i=1}^{m-1} \sum_{j=1}^{n} \left| [\mathsf{mat}(x)]_{i,j} - [\mathsf{mat}(x)]_{i+1,j} \right| . \end{aligned}$$

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$$f(x^k + \alpha(v^k - x^k)) \leq f(x^k) - \alpha G_k + \omega(\alpha D_k), \qquad \texttt{(Curvature)}$$

where $\omega(t) := -t - \ln(1-t)$ for $t < 1$.

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  where $\omega(t) := -t - \ln(1-t)$ for $t < 1$.
- Neither the algorithm nor (`Curvature`) use the special properties of the barrier or the logarithmic homogeneity of f. However, these properties drive our complexity analysis.

Computational Guarantees

Define $\delta_k := F(x^k) - F^*$ for $k \geq 0$ (hence δ_0 is the initial optimality gap)

Define $R_h := \max_{x,y \in \mathsf{dom}\, h} \; |h(x) - h(y)|$ (the variation of h on its domain)

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▷ (Iteration complexity for ε-optimality gap) Let K_ε denote the number of iterations required by gFW-LHSCB to obtain $\delta_k \leq \varepsilon$. Then:

$$K_\varepsilon \leq \lceil 5.3(\delta_0 + \theta + R_h) \ln(10.6\delta_0) \rceil + \left\lceil 12(\theta + R_h)^2 \max\left\{ \frac{1}{\varepsilon} - \frac{1}{\delta_0} \;, 0 \right\} \right\rceil \; .$$

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  - Observe that (Ours) has the exact same dependence on ε as (Kha), namely $O(n^2/\varepsilon)$, but the "fixed" term is slightly inferior to (Kha) by the factor $O(\ln(m/n))$.

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$$\min_{x\in\mathbb{R}^N} \quad \bar{F}(x) := \underbrace{-\sum_{l=1}^N y_l \ln(a_l^\top x)}_{=f(\mathsf{A}x)} + \underbrace{\langle \sum_{l=1}^N a_l, x\rangle + \lambda \mathrm{TV}(x)}_{=h(x)}$$

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  - $\mathrm{TV}(\cdot)$ does not have an efficiently computable proximal operator.
- ▷ However, $\mathrm{TV}(\cdot)$ is a polyhedral function, and the linear-optimization sub-problem

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can be formulated as a relatively simple LP and solved easily using a standard LP solver such as Gurobi.

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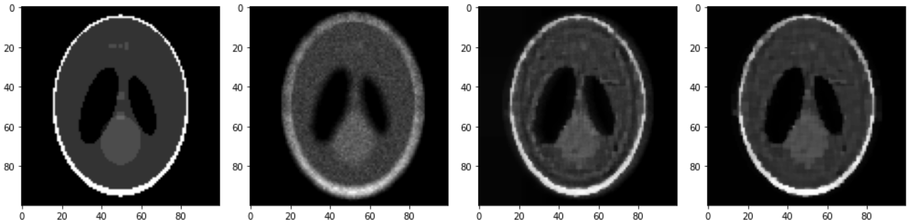
- We chose the starting point $x^0 = \mathsf{vec}(Y)$, and we set $\lambda = 0.01$.
- We used CVXPY to (approximately) compute the optimal objective value $\bar{F}^*$ of (`Deblur`) in order to compute optimality gaps.

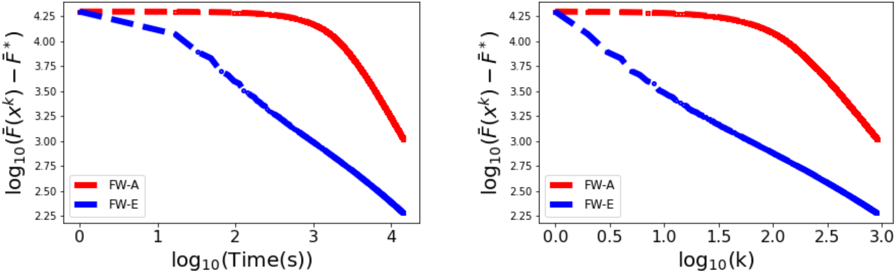
Results: Recovered Images



(a) True image X (b) Noisy image Y (c) `FW-Adapt` (d) `FW-Exact`

Figure 1:
True, noisy and recovered Shepp-Logan phantom image.

Results: Optimality Gaps versus Time and Iterations



(a) Optimality gap versus time (in seconds)

(b) Optimality gap versus iterations

Figure 2:
Comparison of empirical optimality gaps of `FW-Adapt` (`FW-A`) and `FW-Exact` (`FW-E`) for image recovery of the Shepp-Logan phantom image.

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Comparison with Dvurechensky et al. (2020)

- To find an ε-optimal solution, the complexity bound in Dvurechensky et al. (2020) reads:

$$O\left(\sqrt{L(x^0)}D_{\mathcal{X},\|\cdot\|_2}\ln\left(\delta_0/\left(\sqrt{L(x^0)}D_{\mathcal{X},\|\cdot\|_2}\right)\right) + L(x^0)D^2_{\mathcal{X},\|\cdot\|_2}/\varepsilon\right), \qquad \texttt{(Dvu)}$$

where $\mathcal{S}(x^0) := \{x \in \mathsf{dom}\, F \cap \mathcal{X} \,:\, F(x) \le F(x^0)\}$ denotes the initial level-set and

$$L(x^0) := \max_{x\in\mathcal{S}(x^0)} \|\nabla^2 \bar{F}(x)\|_2 < +\infty\ , \text{and}\ \ D_{\mathcal{X},\|\cdot\|_2} := \max_{x,y\in\mathcal{X}} \|x-y\|_2 \quad .$$

- Specialized to the traditional setting, our complexity bound reads:

$$O\big((\delta_0+\theta)\ln(\delta_0) + (\theta)^2/\varepsilon\big). \qquad \texttt{(Ours)}$$

- Our bound (**Ours**) has the following merits:
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  - Norm-invariance

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- Our bound (Ours) has the following merits:
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  - Norm-invariance
  - Interpretability
  - Ease of parameter estimation